MATH FOR OCEANOGRAPHY

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The book of Nature is written in the language of mathematics.

Galileo Galilei

1 Preamble

Mathematics as we humans have conceived it may not be the ultimate language of nature, but it is a darn useful one. At the very least, it's necessary to make sense of most physical oceanography in existence, and even some chemical oceanography. This document is an attempt to make this mathematical formalism less intimidating, by easing gradually into it (instead of the usual, unceremonious onslaught of arcane symbols that usually greets readers of any PO textbook). Yet, there is no denying the reality that this formalism involves a fair amount of vector calculus. It is only useful insofar as each symbol/term is attached to some physics, which this document won't yet do. Be reassured: as you learn to master this ABC, arcane glyphs like ∇ will come to mean something to you, and be synonymous with physical processes for which you have some deep-seated intuition. Patience is advised, as this will happen only gradually. Yet the journey must start somewhere, so without further ado...

2 Differential Calculus

At the quantum level, everything is discrete; however seawater may be usefully regarded as a macroscopic continuum. Consequently, it is amenable to the mechanics of continuous media, which uses differential calculus as its fundamental building block. In this section we refresh[1] your memory about differential calculus.

2.1 Ordinary differentials

2.1.1 Definition

Though Newton is credited with inventing calculus to solve celestial mechanics problems, his work was impossibly abstruse, apparently on purpose. Leibniz helpfully invented the

[1] If the whole concept is new to you, we urge you to rush to the nearest bookstore or MOOC – Khan Academy for instance.

differential notation. This ties the *change* or *dependence* of some quantity, *e.g.* f, to small changes in some independent variable $x \in \mathbb{R}$. If f has value $f(x)$ at x, then after a small but finite increment Δx, $f(x+\Delta x)$ can be written (to first order):

$$f(x+\Delta x) = f(x_0) + S(x)\Delta x + \mathrm{o}(\Delta x^2) \tag{2.1}$$

where $S(x)$ is the slope of the tangent at the point $A = (x, f(x))$ (green line in Fig. 1). The "little o" notation means that there are other terms, but they are all of order $(\Delta x)^2$ or smaller. If we let Δx get close to 0, these terms all vanish, and we arrive at the *derivative*, denoted $f'(x)$:

$$S(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df}{dx}(x) = f'(x) \tag{2.2}$$

What Eq. (2.2) says is that, in a small enough neighborhood of x, the rate of change of f is given by the slope of the tangent to the curve in x (all higher order terms be damned). The limit in Eq. (2.2) will work as long as f doesn't do any funny business as a function of x, like jump around abruptly. The notation $f'(x)$ is common in calculus, but physicists (exceptionally, one might add) are more rigorous when they use $\frac{df}{dx}$. This "differential" notation (due to Leibniz) is clear in three respects:

- it makes clear which variation we are considering, because f could depend on other variables (e.g. y, z, t).
- it makes clear that this is ratio of differentials (small differences).
- it expresses differentiation as an *operator* $\frac{d}{dx}$ acting on a function f, which allows differential operators (Sect. 2.4) to be defined easily, and makes multiple derivatives and the chain rule very easily understood.

That being said, it is more cumbersome than the prime, so most lazy people use f' to denote the derivative. In mechanics and dynamical systems theory, people also use the dot notation: $\dot{u} \equiv \frac{du}{dt}$)

2.1.2 Interpretation

$f'(x_0)$ has a clear geometric interpretations as the slope of the tangent to the curve $\{x; f(x)\}$ at the point $(x_0, f(x_0))$ (Fig. 1). Its physical interpretation is that it reflects the instantaneous rate of change of the function f. The most famous example is that if $u(t)$ is the velocity, then $u'(t)$ is the acceleration.

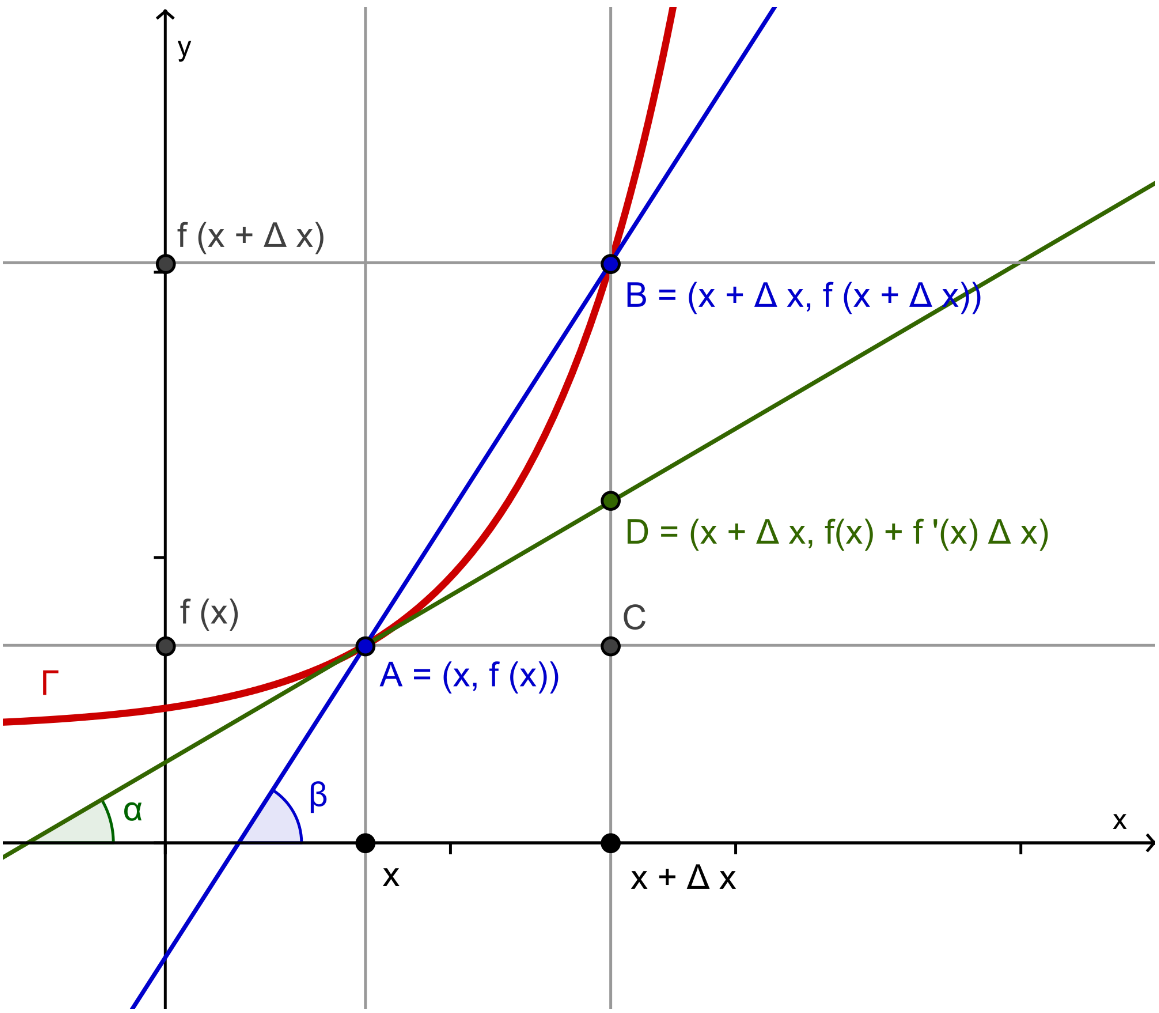

Figure 1: The derivative as a tangent slope (Source:Wikimedia Commons). Note that the first order approximation $f(x) + f'(x)\Delta x$ captures the qualitative behavior of $f(x)$, but is still very far off. To do better, one should include more terms – this is the point of a Taylor expansion.

2.1.3 Properties

If you need to take derivatives of combinations of two or more functions, here called f, g, and h, there are four important rules (with a and b being real constants):

Linearity

$$(af + bg)' = af' + bg'$$

Product rule:

$$(fg)' = f'g + fg'$$

Quotient rule:

$$\begin{aligned} \text{If} \quad f(x) &= \frac{g(x)}{h(x)} \\ \text{Then } f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \end{aligned}$$

Chain rule (inner and outer derivative):

$$\begin{aligned} \text{If} \quad f(x) &= h(g(x)) \\ \text{Then } f'(x) = \frac{df}{dx} &= \frac{dh}{dg}\frac{dg}{dx} = h'(g(x))g'(x) \end{aligned}$$

i.e. derivative of nested functions are given by the outer times the inner derivative.

Based on these simple rules, one can compute the derivative of any closed-form expression! A few useful ones:

$$(e^{\alpha x})' = \alpha e^{\alpha x} \qquad \text{exponential} \tag{2.3a}$$

$$(\ln x)' = \frac{1}{x} \qquad \text{natural logarithm} \tag{2.3b}$$

$$(a_k x^k)' = k a_k x^{k-1} \qquad \text{monomial} \tag{2.3c}$$

By linearity, the last one gives a rule to compute the derivative of any polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$. Recognizing this as a sum of monomials, we get:

$$P'(x) = \sum_{k=1}^{n} k a_k x^{k-1} \tag{2.4}$$

Notice how P was of degree n, but P' is of degree $n-1$. Successive derivatives would lower the order even further. For instance the second derivative (linked to curvature) is:

$$P''(x) = \sum_{k=2}^{n} k(k-1) a_k x^{k-2}$$

This will become important in Sect. 2.2.2

2.1.4 Oceanographic application: radioactive decay

Derivatives are cool, but they are, by themselves, of limited use. Their ubiquity in the natural sciences comes from the fact that most problems in a continuum are stated in terms of equations that involve a function and one or more of its derivatives: we call those *differential equations*. As said earlier, this is what motivated Newton to invent differential calculus in the first place (celestial mechanics is nothing if not a differential equations smörgasbord of cosmic proportions). Let us bring this back to salty water: the most elementary form of a budget for any chemical or biological species ξ is an equation of the form:

$$\frac{d\xi}{dt} = \mathrm{Input}_{\xi} - \mathrm{Output}_{\xi} \tag{2.5}$$

For examples, see *Sarmiento and Gruber* (2006, chapter 1). Let's say $\xi = [^{14}\mathrm{C}]$. Assuming the system is isolated (*e.g.* you're interested in a water parcel that hasn't seen the atmosphere in a few years), the input term is 0, and the output (*i.e.*, the rate of loss) is proportional to the amount of $^{14}\mathrm{C}$, so we write:

$$\mathrm{Output} = \lambda[^{14}\mathrm{C}]$$

where $\lambda > 0$ is the radioactive decay constant, which is related to the *half-life* as $\tau_{1/2} = \frac{\ln 2}{\lambda}$, with $\ln$ the natural logarithm. In the case of $^{14}\mathrm{C}$, $\tau_{1/2} \simeq 5730y$, so $\lambda \simeq 1.2 \times 10^{-4}\ \mathrm{yr}^{-1}$. This would lead you to write a budget like this:

$$\frac{d[^{14}\mathrm{C}]}{dt} = -\lambda[^{14}\mathrm{C}]$$

The general form of this equation is thus:

$$\frac{d\xi}{dt} = -\lambda\xi \tag{2.6}$$

There are a myriad processes that work this way: the decay of radioactive elements in a crystal lattice, the decay of a metabolite in the bloodstream, the relaxation of a thermally isolated system towards equilibrium, the extinction of light in an absorbing medium (think: sunlight in the upper ocean), the decay of pressure with altitude, or the growth of money on a savings account with fixed interest rate.

Eq. (2.6) is known as an *ordinary differential equation*: it is an equation that involves the function $\xi(t)$ and at least one of its derivatives, but unlike the partial differential equations we'll see later, it is *ordinary* in the sense that there is only one independent variable (time t, in this case, though it could really be anything). Solving a differential equation means coming up with an algebraic expression (ideally closed-form) for $\xi(t)$.

Luckily, no fancy math is required to solve this one ; it can be done by a method called **separation of variables**, where ξ and t are placed on each side of the equation:

$$\frac{d\xi}{dt} = -\lambda\xi \Rightarrow \frac{d\xi}{\xi} = -\lambda dt$$

Next, we integrate both sides of the equation from 0 to t:

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\xi} = -\int_0^t \lambda dt$$

We recognize that the integral on the left will involve the natural logarithm, $\ln x$, which is the anti-derivative of $1/x$[2], so the result is:

$$\ln \xi - \ln \xi_0 = -\lambda(t - 0)$$

Where $\xi_0 = \xi(t = 0)$ is the initial value of ξ. Rearranging, we get $\ln\left(\frac{\xi}{\xi_0}\right) = -\lambda t$. After exponentiating both sides (since $\exp$ is the reciprocal function of $\ln$), we get the coveted result:

$$\xi(t) = \xi_0\, e^{-\lambda t} \tag{2.7}$$

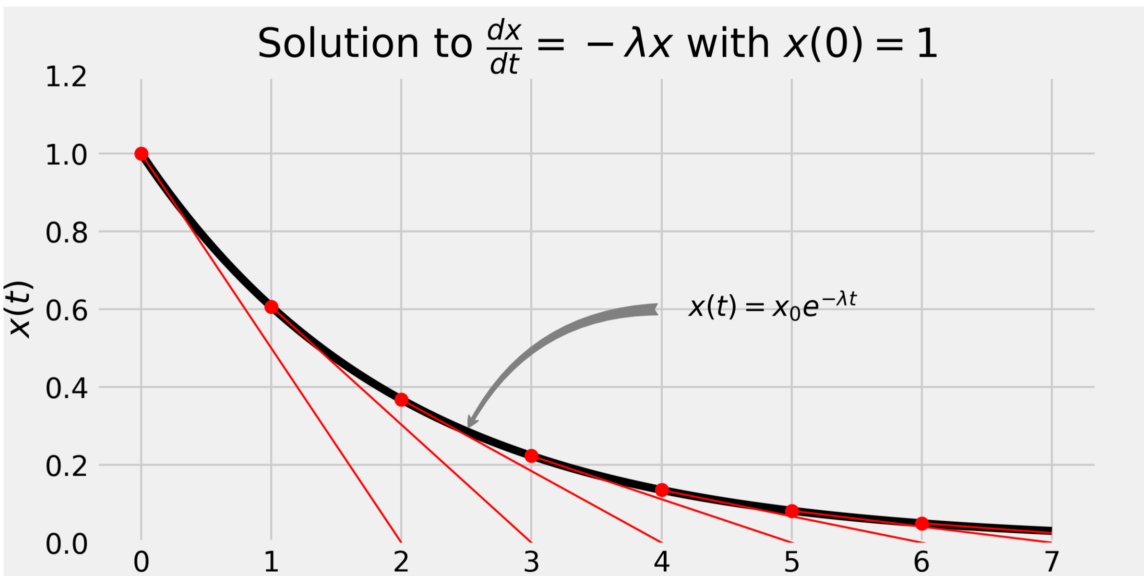

Figure 2: Exponential decay. Note how the slope of the tangents (red lines) progressively tapers off to 0 as x decreases.

This is a textbook case of exponential decay, and it is sketched in Fig. 2, where the tangents are highlighted at various points. Notice how the slopes are initially steep, then weaken as time progresses. This we could have guessed from Eq. (2.6), which literally says "the rate of change is always negative (decay), and it is steepest when ξ is largest". We see that the exponential solution satisfies this requirement ; in fact, it is the *only* solution that does (*i.e.*it is unique), and furthermore, this gives us a clue as to why exponentials are so ubiquitous in the physical and social sciences. Quite simply, anytime the rate of change of something depends linearly on the amount of that thing, then the solution is likely to involve an exponential of some kind.

Stepping back, we see that solving differential equations requires not only finding the functional form of the solution (here, a decaying exponential) but also supplying a number of initial and/or boundary conditions (it will become clear in a moment what we mean by that). In general, differential require as many such values (constants) as the order of the

[2]That is, $(\ln x)' = \frac{1}{x}$

equation: this is a first-order equation because it only involves the first derivative $\xi'(t)$, so only one such value (ξ_0) is required. Because all that was needed to solve this one was the initial value, Eq. (2.6) is known as an *initial value problem*. An examples of oceanographic *boundary value problem* is to compute the distribution of inert gases like chloroflurocarbons (CFCs) given a known input of the gasses from human emissions into the atmosphere: we know from atmospheric inventories how much that is, and the goal is so see how those chemicals invade the ocean over time, starting at the surface.

In general, most problems involve both initial and boundary conditions. Indeed, Eq. (2.6) could be generalized to cases where there is an input term as well (say a time-dependent function $F(t)$, i.e. a forcing, which is a form of boundary condition), and you can check any good book on differential equations for that. But time is of the essence, so let's now turn our attention to partial differentials.

2.2 Partial Differentials

Ordinary differentials are all we need when we consider variations as a function of a single *independent* variable (time, in the case above). However, in nature, things usually vary along 3 dimensions of space, as well as time, so in principle we need to represent functions accordingly, for instance, the West-East velocity of a water parcel should be written $u(x, y, z, t)$. That's 4 variables, and potentially many ways of changing. Partial differentials is what we use to quantify that.

2.2.1 Definition

A *partial differential* is a differential just like Eq. (2.2), but varying *only one variable at a time, while keeping the others constant*. To distinguish it from an ordinary differential, we use the curly d-like symbol ∂, defined as follows:

$$\left.\frac{\partial u}{\partial t}\right|_{x,y,z} = \lim_{\Delta t \to 0} \frac{u(x, y, z, t + \Delta t) - u(x, y, z, t)}{\Delta t} \tag{2.8}$$

Similarly, if we want the variation along x, we would hold y, z, t constant, and look at the infinitesimal rate of change along the x axis:

$$\left.\frac{\partial u}{\partial x}\right|_{y,z,t} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y, z, t) - u(x, y, z, t)}{\Delta x} \tag{2.9}$$

and so on and so forth for $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$. It is understood that we only vary one variable at a time, so the bracket term on the right is usually dropped. Further, to save some typing, some people use the notation $\frac{\partial u}{\partial y} \equiv \partial_y u \equiv u_y$. Things get more interesting once we start differentiating multiple times. The second partial derivative in y would be $\frac{\partial^2 u}{\partial y^2} \equiv \partial^2_{yy} u \equiv u_{yy}$. You can also have mixed derivatives like $\frac{\partial^2 u}{\partial x \partial y} \equiv \partial^2_{xy} u \equiv u_{xy}$. And so on and so forth.

Given the similarity in notation between $\frac{\partial}{\partial t}$ and $\frac{d}{dt}$, it should come as no surprise that they are related. Formally, this is done via the so-called *total differential*:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt \tag{2.10}$$

This states that the total variation in the function u in the neighborhood of the point (x, y, z, t) is simply the sum of the partial derivatives times the variations in the coordinates. This will become important later when we discuss the Lagrangian (or material) derivative.

Exercise: verify that for a function of a single variable, the total and partial derivatives are the same.

2.2.2 Oceanographic Application: the seawater equation of state

To quote one of my grad school teachers, Prof. Marc Spiegelman: "gravity acting on density differences is 90% of what happens on this planet". The density of seawater (ρ) is therefore a key oceanographic variable, whose even minute variations (between 1024 and $1029\ \mathrm{kg m^{-3}}$) are enough to explain the motions of planetary circulations like the Atlantic Meridional Overturning Circulation.

It is known that ρ depends only on temperature, the amount of solute (mostly salt, though dissolved silica can play a role), and pressure[3]. We call this the *seawater equation of state*, which we write:

$$\rho = \rho(T, S, p) \tag{2.11}$$

Unlike dry air, whose equation of state is given by a very simple relation (the ideal gas law[4]), the functional form of the seawater equation of state is immensely complicated, and can only be approximated by a polynomial. Indeed, the latest attempt at defining the thermodynamic properties of seawater (TEOS-10) includes up to 75 terms! In fact, no textbook I have seen ever bothers to write this those interminable expressions; nowadays, people simple provide code to compute them[5]. Yet, a useful approximation is to express this as first order deviations around a base state $\rho_0 = \rho(T_0, S_0, 0)$, where $T_0 = 20°\mathrm{C}$, $S = 34.7$, and $p = 0$ (surface):

$$\rho = \rho_0 \left[1 - \alpha_T(T - T_0) + \beta_S(S - S_0) + k_p p + R_2(T, S, p)\right] \tag{2.12}$$

[3]The precise definition of those can be gnarly; there are many different ways to express the equation of state, depending on which variables you choose. The current wisdom is to use *conservative temperature* Θ and *absolute salinity* S_A, but for the purpose of this math writeup, we stick to the more usual *in situ* temperature T and salinity S.

[4]$p = \rho R_d T$, where $R_d = 287\ \mathrm{JK^{-1}kg^{-1}}$

[5]https://www.teos-10.org/software.htm

where:

- $\alpha_T = -\left(\frac{\partial \rho}{\partial T}\right)_{S_0,0}$ is the thermal expansion coefficient
- $\beta_S = +\left(\frac{\partial \rho}{\partial S}\right)_{T_0,0}$ is the haline compression coefficient
- $k_p = +\left(\frac{\partial \rho}{\partial p}\right)_{T_0,S_0}$ is the compressibility
- $R_2(T, S, p)$ is a residual term with higher order derivatives, including cross-terms like ∂_{TS} or ∂_{Sp}. Essentially, this terms gathers all the other terms in a residual of order 2 or more.

These coefficients are defined here so they are all positive: increasing salt content and pressure increase density, so these terms appear with a $+$ in Eq. (2.12); conversely, higher temperature decreases density, so this term appears with a $-$. While conventions vary by author, what matters is that these definitions show how the coefficients in any polynomial representation of Eq. (2.11) may be obtained by partial differentiation of the full expression. This is made possible by Eq. (2.4), but no-one in their right mind ever does this by hand; the good news is that it's all coded up for you in a toolbox like GSW, available in all major programming languages[6].

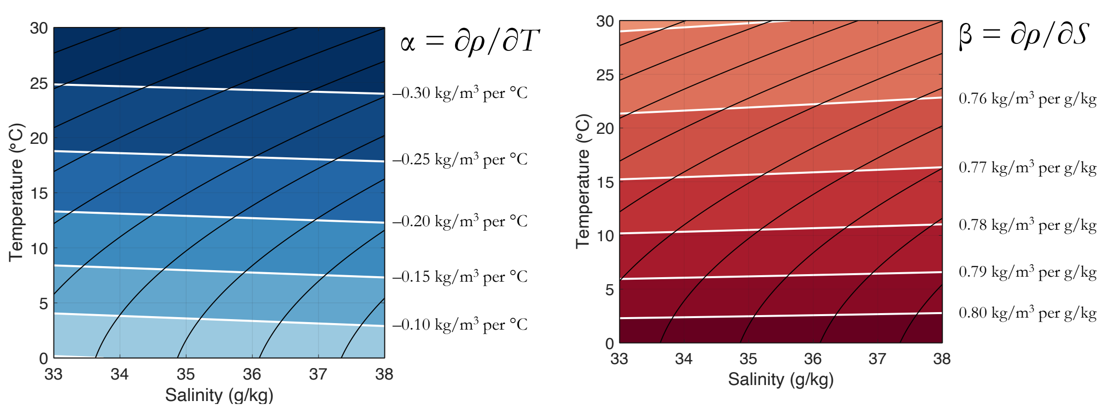

Figure 3: Dependence of seawater density ρ on temperature and salinity. Credit: Kris Karnauskas, CU Boulder.

The bad news is that these coefficients are not actually constant. If you plot lines of equal density (known as *isopycnals*) in a temperature-salinity diagram, you will notice that they are curved (Fig. 3); this implies that β_S is higher at low temperatures than high temperatures. We also see that isopycnal contours are more spread out in the lower right corner (cold, saline) than in the upper left (warm and fresh), so α must change with T and S as well. This obviously would not be possible if the terms lumped into $R_2(T, S, p)$ were actually negligible. Clearly we can't get away with a linear approximation, at least not

[6] https://www.teos-10.org/software.htm

everywhere. Does this mean that we need all 75 terms of TEOS-10? Can we isolate which terms actually matter to represent the essential physics? This question was explored by *Roquet et al.* (2015), who identified the terms that matter for the three important deviations from nonlinearity:

Cabbeling The thermal expansion coefficient depends to first order on temperature, an effect that is commonly referred to as cabbeling. One consequence is that by mixing two water masses, you can generate a third water mass that is denser than either (seeFig. 4, left). To capture this effect, you need a quadratic term in T(strictly speaking, in the Conservative Temperature Θ^2(*Roquet et al.*, 2015)).

Thermobaricity The thermal expansion coefficient also depends critically on depth, through the thermobaric effect. This means that seawater is more sensitive to temperature (greater α_T) at high depth (pressure), an effect that becomes important for the very densest water masses like Antarctic Bottom Water. To capture this effect, you need a term proportional to ΘZ (*Roquet et al.*, 2015).

Freezing temperature It turns out that salinity depresses the freezing point of seawater, enabling water to stay liquid at negative temperature (see Fig. 4, right. pro tip: you can use that chilling fact to make ice cream!). To capture this effect, you need a second-order equation of state with quadratic terms in (Θ, S_A, Z) as well as cross-terms

To incorporate all these physical effects, *Roquet et al.* (2015) found that a second-order expansion was sufficient:

$$\rho(\Theta, S_A, Z) = -\frac{C_b}{2}(\Theta - \Theta_\circ)^2 - T_h Z\Theta + b_\circ S_A \tag{2.13}$$

where

$$\begin{aligned}
C_b &= 0.011\ \mathrm{kgm^{-3}K^{-2}} \text{ sets the sensitivity of thermal expansion to temperature} \\
T_h &= 2.5\times 10^{-5}\ \mathrm{kgm^{-4}K^{-1}} \text{ gives the sensitivity of thermal expansion to depth} \\
b_\circ &= 0.77\ \mathrm{kgm^{-3}(g.kg^{-1})^{-1}} \text{ is the constant haline contraction coefficient} \\
\Theta_\circ &= -4.5°\text{C is the temperature at which surface thermal expansion is 0}
\end{aligned}$$

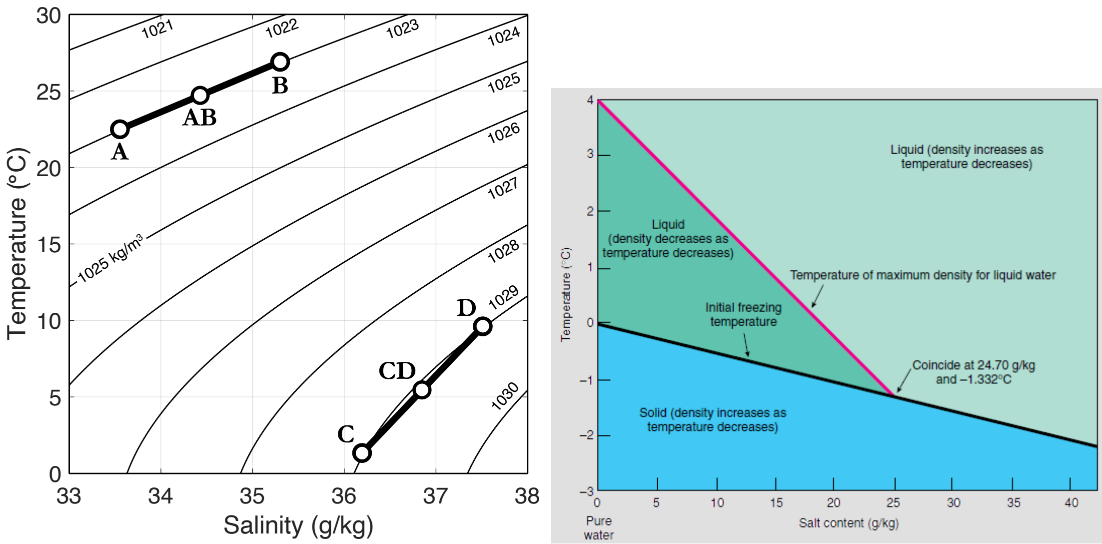

Figure 4: Left: impact of isopycnal curvature on seawater mixing: C and D start out at the same density, but mixing them creates a water mass CD with intermediate T and S, yet higher density. The effect is barely noticeable for A, B and AB because they lie on an isopycnal with little curvature (credit: Kris Karnauskas). Right: Effect of salinity on the freezing temperature of seawater, and the temperature of maximum density. For $S > 24.7$ (most of the ocean), maximum density and freezing temperature coincide, but they diverge at low salinities. For freshwater, maximum density is reached around 4°C. (credit: Wikipedia)

2.2.3 Oceanographic Application: the Gradient

An incredibly important notion in physics is that of *gradient*: informally, anyone who's ever played with crayons knows that it has to do with the smooth variation of a continuous variable across space. In turns out that this notion can be formalized via a differential *operator*, called the gradient, denoted ∇. Imagine that you have a function ψ that depends on spatial coordinates (x, y, z). Its total differential is thus (recall Eq. (2.10)):

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial z}dz \tag{2.14}$$

If we define an element position vector $d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

then the gradient of ψ is the *vector*

$$\boldsymbol{\nabla}\psi = \begin{pmatrix} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial y} \\ \frac{\partial \psi}{\partial z} \end{pmatrix} \tag{2.15}$$

where $\boldsymbol{\nabla}$ is called "nabla", or sometimes "del" (in relation to the letter Δ). Recognizing

Eq. (2.14) as a dot product[7], we may write:

$$d\psi = \boldsymbol{\nabla}\psi \cdot d\vec{r}$$

By definition, this vector always points to the highest values of ψ, like a compass. Let us take an example in 2D to drive home that point. Fig. 2.2.3 shows a fictional anomaly in sea-level or dynamic height depending on "longitude" x and "latitude" y, as well as its gradient ∇h. One can see that the arrows indeed point to the various maxima: in the center they point toward the origin, where the global maximum is located. In the outskirts, they point towards the secondary maximum located in the outer rim.

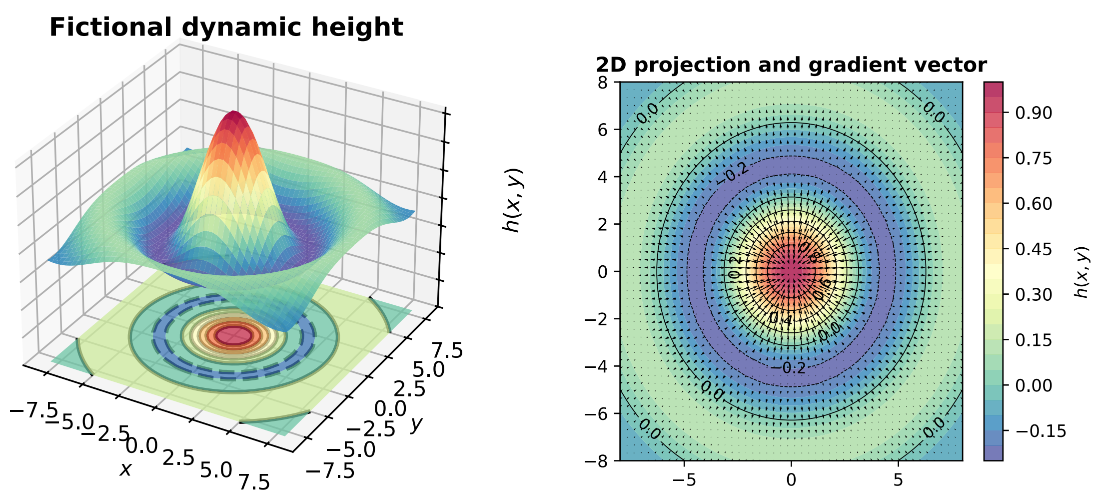

Figure 5: Left: a fictional anomaly in sea-level or dynamic height $h(x, y)$. Right: the same anomaly $h(x, y)$, plotted as contours, and its gradient ∇h, plotted as arrows. Notice how the arrows point to the nearest maximum, and their length (norm) is proportional to the slope of the surface.

Now that we have these basic building blocks of partial derivatives, we can start to use them to define very useful (albeit abstract) mathematical objects. Before we do that, we need to define our coordinates.

[7]If you need a refresher on linear algebra, check out my book, Appendix B

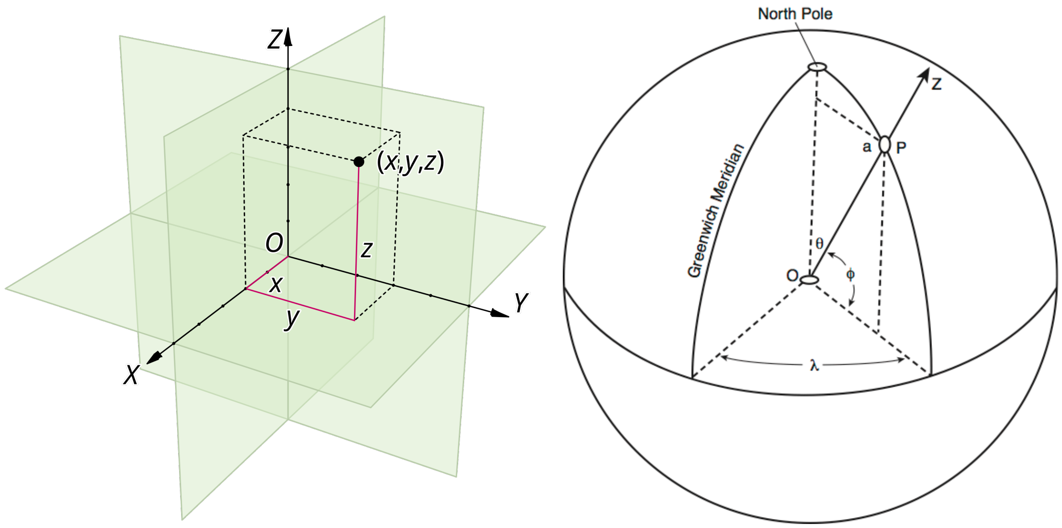

Figure 6: **Two ways to represent space:** left: Cartesian coordinate system (Credit: Wikipedia). right: Spherical geometry as applicable to oceanography (Credit: *Wunsch* (2015))

2.3 Coordinate Systems

You may be aware that there is no mandatory choice of coordinates; as far as classical physics are concerned, space is a 3-dimensional manifold, so one is free to use any 3 vectors that span the space and assign coordinates along those vectors. In practice, there are two useful ones we'll use in this class:

Cartesian coordinates are the classical (x, y, z), shown in Fig. 6 (left).

Spherical Coordinates use a radius r and two angles in radians[8], as shown in Fig. 6 (right). In the atmosphere and ocean sciences, those would be longitude (commonly denoted λ) and latitude (commonly denoted ϕ), though other choices are possible. Geophysicists often prefer to use the *co-latitude* $\theta = \frac{\pi}{2} - \phi$. To go between coordinate systems, we can use the relations:

$$
\begin{array}{ll}
r = \sqrt{x^2 + y^2 + z^2} & x = r\cos(\lambda)\cos(\phi) \\
\lambda = \arctan(y, x) & y = r\sin(\lambda)\cos(\phi) \\
\phi = \arcsin\left(\frac{z}{r}\right) & z = r\sin(\phi)
\end{array}
\qquad (2.16)
$$

The choice of coordinates does affect the expression of partial derivatives. For instance,

[8] 1 radian = $\frac{180}{\pi}$ degrees.

the gradient in spherical coordinates is:

$$\nabla\psi(r,\lambda,\phi) = \begin{pmatrix} \frac{\partial\psi}{\partial r} \\ \frac{1}{r\cos(\phi)}\frac{\partial\psi}{\partial\lambda} \\ \frac{1}{r}\frac{\partial\psi}{\partial\phi} \end{pmatrix} \tag{2.17}$$

A good rule of thumb is that a spatial derivatives in spherical coordinates always have to include the radius somewhere to maintain units of length^{-1}. A $\frac{\partial}{\partial\lambda}$ or $\frac{\partial}{\partial\theta}$ term will always be scaled by $\frac{1}{r}$, usually with some kind of trigonometric factor involving a sine or cosine of latitude.

Spherical coordinates are clearly the more natural choice on a rounded planet like ours, but as the previous expression demonstrates, they are a bit of a pain to deal with. As much as possible, we'll work with Cartesian coordinates. One important consequence of spherical coordinates is for integration (e.g. to compute a global flux): as meridians converge towards the poles, the area of an element between λ and $\lambda + d\lambda$ and ϕ and $\phi + d\phi$ shrinks to near 0. Indeed, its area may be expressed as:

$$d\mathcal{A} = \cos\phi d\lambda d\phi \tag{2.18}$$

Thus, for many computations it is common to scale fields by the cosine of latitude to ensure geometric accuracy. From this rule, you can show that fully half of the surface area of the planet lies with $30°$: the tropics rule, and most map projections misleadingly show the opposite. You may play with such projections here.

2.4 Differential Operators: Div, Grad Curl and all that

It may have gently passed you by, but something rather extraordinary just happened in Sect. 2.2.3: we took a scalar function ψ and we turned it into a 3D vector. Indeed the gradient is one of the most useful *differential operators* in existence: it is a mathematical object that can take another object (say, a scalar field like temperature or oxygen concentration) and turn it into a 3D vector. This operator serves as the foundation for other such differential operators, the divergence (Div), the rotational (Rot, or Curl), and the Laplacian, which are an essential component of how physics expresses conservation laws. To demystify the equations of motion, let's look at those operators.

2.4.1 Gradient

In Cartesian coordinates, this operator may be written in the abstract form:

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{2.19}$$

That way, we could say that Eq. (2.15) was simply the *gradient operator* ∇ applied to the scalar function ψ – we write $\nabla\psi$ for short. We shall see shortly why that notation is useful.

2.4.2 Divergence

Definition We call *divergence* the operator $\boldsymbol{\nabla}\cdot$, which turns a vector $\boldsymbol{u} = (u_x, u_y, u_z)^\top$ into a scalar. In Cartesian coordinates x, y, z, this is:

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{2.20}$$

Those among you who've studied a bit of geometry will recognize that this is the inner (dot) product of $\boldsymbol{\nabla}$ and $\boldsymbol{u}$ (Fig. 8, left). The divergence is so named because it measures the tendency of a vector field to spread out ($\boldsymbol{\nabla}\cdot\boldsymbol{u} > 0$) or converge ($\boldsymbol{\nabla}\cdot\boldsymbol{u} < 0$). If $\boldsymbol{u}$ is the velocity of a fluid of uniform density $\rho_\circ$, then $\boldsymbol{\nabla}\cdot(\rho_\circ\boldsymbol{u})$ is exactly the net accumulation of mass in a given volume dV over a unit time dt. This may be established using Gauss's divergence theorem, which states that the net flux through a surface S is equal to the volume integral of this divergence operator:

$$\iint_S \boldsymbol{u}\cdot d\boldsymbol{N} = \iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{u}dV \tag{2.21}$$

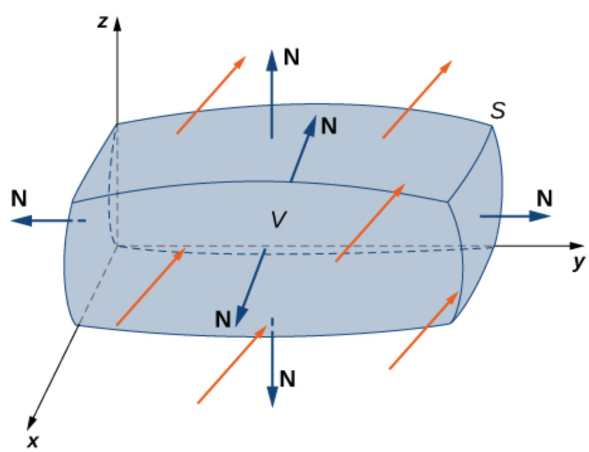

Figure 7: Illustration of divergence out of a control volume V bounded by a surface S. Credit

This is what is used to transform 2D integrals into 3D integrals, and *vice versa*. What do you mean "I couldn't care less"? Let me show you why you should care, by using this theorem to track the mass budget in a region.

Oceanographic application: budgets Say we are trying to see how much mass flows in and out of the control volume V (Fig. 7). The rate of change of the mass M (the volume integral of the density ρ) has to equal how much mass is flowing in and out of the surface S. The first can be expressed as the volume integral of density; the second is the surface integral of the mass flux $\rho\boldsymbol{u}$ all around S:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\iiint_V \rho dV = -\iint_S \rho\boldsymbol{u}\cdot d\boldsymbol{N}$$

Note that the minus sign is necessary because the normal vector $\boldsymbol{N}$ points everywhere outward (it counts positively what is leaving the blob), whereas we count positively mass going into the blog, so we need to flip sign to get our accountancy right. By virtue of the divergence theorem Eq. (2.21), the right-hand side of the previous expression may be rewritten:

$$\frac{\partial}{\partial t} \iiint_V \rho dV = - \iiint_V \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} dV \tag{2.22}$$

The amazing thing about Eq. (2.22) is that it is valid for any volume V, so we are allowed to shrink this to an elementary volume dV; that is, we dispense with those pesky integrals altogether:

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

or, moving everything to the left side:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \tag{2.23}$$

This is no more, no less than the conservation of mass in an infinitesimal volume element dV, obtained here by applying the divergence theorem to the mass flux $\rho \boldsymbol{u}$. This turns out to form the basis for all conservation equations in a fluid. In fact, if you need to track the concentration of n conservative tracers $c_1, c_2, \cdots, c_n$ (no source terms, no diffusion), all you would have to do is write n such budgets:

$$\begin{aligned}
\frac{\partial (\rho c_1)}{\partial t} &+ \boldsymbol{\nabla} \cdot (\rho c_1 \boldsymbol{u}) &= 0 \\
\frac{\partial (\rho c_2)}{\partial t} &+ \boldsymbol{\nabla} \cdot (\rho c_2 \boldsymbol{u}) &= 0 \\
\vdots & \qquad \vdots & \vdots \\
\frac{\partial (\rho c_n)}{\partial t} &+ \boldsymbol{\nabla} \cdot (\rho c_n \boldsymbol{u}) &= 0
\end{aligned}$$

Don't go anywhere, there are a couple more operators we need.

2.4.3 Curl (Rotational)

If the divergence proceeds from an inner (dot) product (smooshing two vectors into a scalar), the curl (or rotational) proceeds from an *outer product*, also known as a cross product (Fig. 8, right). This takes two vectors in a plane and sending them into a third vector normal to that plane.

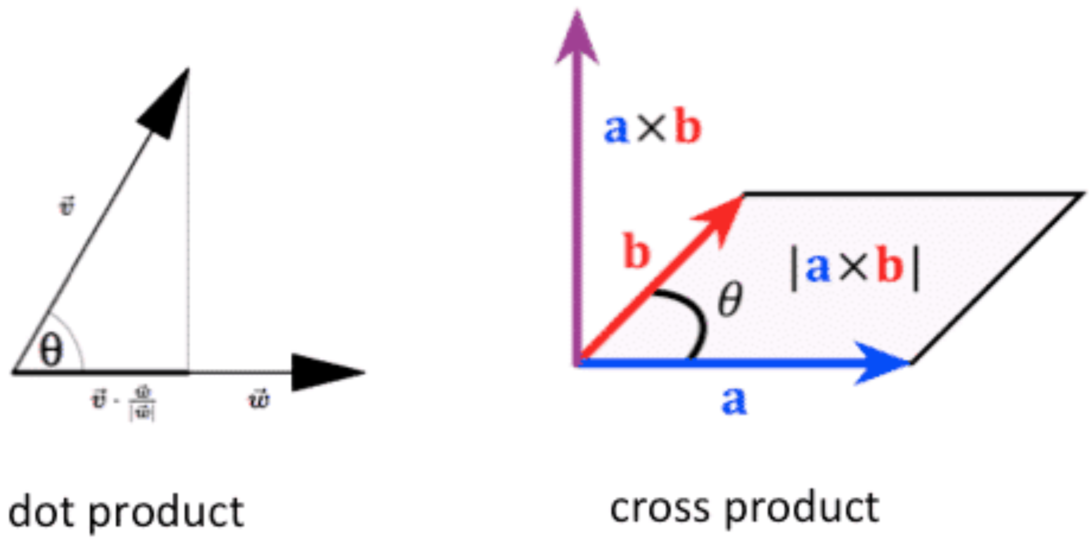

Figure 8: Inner vs Outer product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The inner product maps $\boldsymbol{a}$ and $\boldsymbol{b}$ to a scalar $\|a\|\|b\|\cos\theta$, while the outer product maps $\boldsymbol{a}$ and $\boldsymbol{b}$ to a vector $\boldsymbol{c}$ along the normal to the plane defined by $\boldsymbol{a}$ and $\boldsymbol{b}$, whose norm is proportional to the area of the parallelogram defined by $\boldsymbol{a}$ and $\boldsymbol{b}$. Source

For a vector $\boldsymbol{u} = (u_x, u_y, u_z)^\top$:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\boldsymbol{\imath}} - \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{\boldsymbol{k}}$$

In oceanography, the Curl is used to define an extremely important quantity called *vorticity*, which is key to understanding many things, for instance westward intensification. Of particular interest is the projection of $\boldsymbol{\nabla} \times \boldsymbol{u}$ along the local vertical $\hat{\boldsymbol{k}}$, called "relative vorticity":

$$\zeta = \hat{\boldsymbol{k}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \tag{2.24}$$

2.4.4 Laplacian

The Laplacian, denoted ∇^2, is formally defined as the divergence of the gradient: $\nabla^2\psi = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi)$. Applying the standard rules of the dot product, its expression in Cartesian coordinates is:

$$\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.25}$$

That is, it is the sum of the second derivatives in space – a measure of curvature. In spherical coordinates, We will see this operator play a central role in wave propagation and diffusion. As a slight abuse of notation, the Laplacian can also apply to vectors (even tensors!), like $\boldsymbol{u} = (u_x, u_y, u_z)^\top$:

$$\boldsymbol{\nabla}^2\boldsymbol{u} = \begin{pmatrix}\nabla^2 u_x\\ \nabla^2 u_y\\ \nabla^2 u_z\end{pmatrix} \tag{2.26}$$

The Laplacian occurs in nearly all the partial differential equations we study in oceanography (Sect. 3). It is far less friendly in spherical coordinates, so better build intuition in Cartesian coordinates and pick up any good fluid dynamics or vector calculus book for the expression on a sphere.

2.4.5 Identities

Now let's play with those operators! There are a lot of identities involving these operators, which are summarized here. Here are the most useful ones for our purposes:

A gradient does not rotate:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = 0 \text{ for any scalar field } \psi \tag{2.27}$$

A curl is non-divergent:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}) = 0 \text{ for any vector } \boldsymbol{u} \tag{2.28}$$

Curl of curl:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}^2\boldsymbol{u} \tag{2.29}$$

Advection identity:

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{u}|^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{u}|^2 + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} \tag{2.30}$$

One of the coolest consequences of the first two identities is that one can represent the velocity field of the ocean as the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field. In fact, the irrotational field can be represented as the gradient of a potential Φ, and the solenoidal field can be represented as the curl of another field $\mathbf{\Psi}$ so we get:

$$\boldsymbol{u} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla} \times \mathbf{\Psi} \tag{2.31}$$

This is called a Helmholtz decomposition, and will be very useful later. In the vast majority of the ocean and the atmosphere, the flow is very nearly incompressible (non-divergent) so the solenoidal field dominates. In 2D , the term $\boldsymbol{\nabla} \times \mathbf{\Psi}$ can be reduced to $\hat{\boldsymbol{k}} \times \boldsymbol{\nabla}\Psi$, where Ψ is a scalar function, called a *streamfunction*, whose isolines define the flow (Fig. 9). It is most convenient because it allows to characterize the flow by one function instead of a vector. Perhaps many of you have stared at streamfunctions all your life without knowing where they came from. Well, you have Helmholtz's theorem to thank for them.

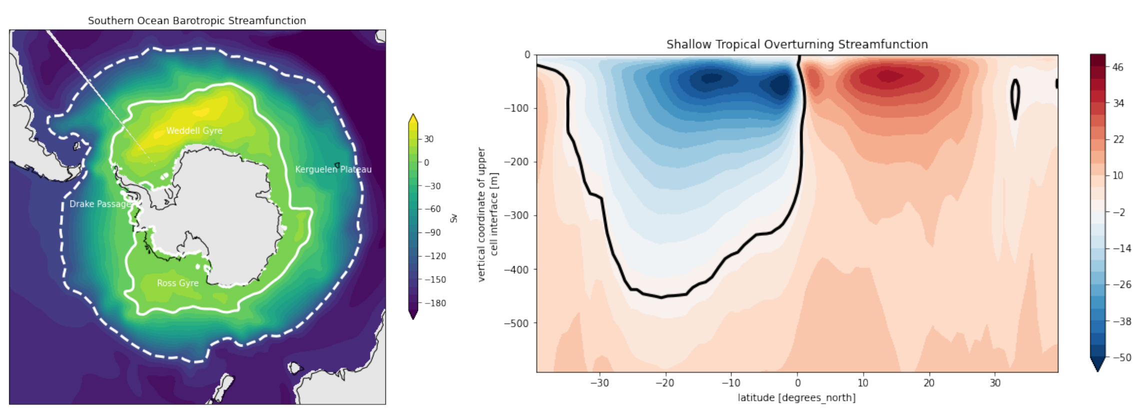

Figure 9: **Streamfunctions in physical oceanography:** Left: barotropic (i.e. vertically integrated) streamfunction over the Southern Ocean. Positive values indicate an anticyclonic circulation (clockwise in the Southern Hemisphere), negative contours depict an cyclonic circulation (clockwise). Right: overturning (longitudinally-integrated) streamfunction in the upper 500m of the global ocean, highlighting the subtropical cells, with upwelling near the equator and downwelling (subduction) in the tropics. Data credit: ECCO state estimate. Figure credit: R. Abernathey

3 Notable Partial Differential Equations

The operators defined above are useful shorthands that lighten the notation of many equations in physics, be it electromagnetism, fluid mechanics, plasma physics, seismology or astrophysics. The resulting equations are part of a category called *Partial Differential Equations* (PDEs), because they involve the symbol ∂. Like other differential equations, we define their *order* as the degree of the highest derivative: for instance, if a $\frac{\partial^3}{\partial x^3}$ term occurs, then it is a 3-order equation (same with other variables).

It is important to state that most PDEs are so complicated that only a computer can solve them (and then, only approximately). However, there are a handful of PDEs that you should know about, because they serve to build intuition both within and beyond oceanography.

3.1 The Heat (or Diffusion) Equation

3.1.1 General Form

One of the most useful PDES in all of physics, and particularly oceanography, is the diffusion equation. Imagine you have a tracer with concentration $\chi(x, y, z, t)$. Fick's law states that molecular diffusion induces a flux $\vec{q}_\chi$ going downgradient, and proportional to this gradient:

$$\vec{q}_\chi = -D\boldsymbol{\nabla}\chi \tag{3.1}$$

Where D is a constant diffusivity (in $\mathrm{m^2 s^{-1}}$). The minus sign here is essential: remember that the gradient points to the nearest maximum, whereas clearly any flux must be from areas of relative abundance (high χ) to areas of relative scarcity (low χ). The minus sign turns the gradient around from high to low.

The conservation of mass (see Sect. 2.4.2) then mandates:

$$\frac{\partial \chi}{\partial t} + \boldsymbol{\nabla}\cdot \vec{q}_\chi = 0$$

By definition of the Laplacian Sect. 2.4.4, the second term is:

$$\boldsymbol{\nabla}\cdot(-D\boldsymbol{\nabla}\chi) = -D\nabla^2\chi$$

So the equation becomes:

$$\frac{\partial \chi}{\partial t} = D\nabla^2\chi \tag{3.2}$$

This is the standard form for a diffusion equation with constant diffusivity. Because the Fickian form Eq. (3.1) applies to many variables/processes (e.g. the flow of heat in a continuous medium, the flow of electric charge in a conductor, the flux of mass eroded from a sloping landscape), this form applies to an astonishing number of processes, in many areas of physics, mathematics and finance.

In many cases it is rather useful to look at a 1-dimensional problem in the vertical, neglecting horizontal variations in x and y:

$$\frac{\partial \chi}{\partial t} = D\frac{\partial^2 \chi}{\partial z^2} \tag{3.3}$$

Now we are down to 2 variables: time and depth. Because the concentration χ (or temperature, electrical charge, or other variable being diffused) appears on both sides, it effectively disappears, and the equation binds the length scale ℓ to the time scale τ. Specifically, one can write $D \sim \frac{\ell^2}{\tau}$, which means that after a time τ, a perturbation in χ has traveled a distance $\ell \sim \sqrt{D\tau}$.

This suggests that the solution might depend on a self-similarity variable $\eta = \frac{z}{2\kappa t}$. Indeed, one can use this strategy to transform this 2D partial differential equation into a 1D ordinary differential equation, whose solution involves a Gaussian integral[9]. One gets the same solution by the Fourier transform. In both cases, if a pulse of high concentration is injected at depth $z_\circ$ at time $t = 0$, then χ varies according to the Gaussian form:

$$\chi(z,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left[-\frac{(z-z_\circ)^2}{4Dt}\right] \tag{3.4}$$

This is illustrated in Fig. 10, which shows how the action of diffusion is to spread out the initial spike, until eventually all concentration is uniform. However, the total mass of the substance, that is the integral $M(t) = \int_{-\infty}^{+\infty} \chi(z,t)dz$ is a constant (unity, in this case).

3.1.2 Oceanographic Application: estimating diffusivity

The wildest thing about this simple theory is that it actually works! The idealized physics of 1-dimensional diffusion can be observed in our very real, messy, 3-dimensional ocean. In February 2009, as part of the DIMES campaign, *Watson et al.* (2013) released "76 kg of trifluoromethyl sulphur pentafluoride (CF_3SF_5) [...] about 2,000 km upstream of the Drake Passage, in the Antarctic Circumpolar Current (ACC) between the Subantarctic and Polar Fronts", at a depth of approximately 1,500m. Subsequent field campaigns (Fig. 11A) within the ACC then measured the vertical distribution of this man-made tracer, whose natural concentration is 0. The measured profiles are shown in Fig. 11B; the parallel with Fig. 10 is staggering, and should have you lose your mind a little bit. More to the point, these observations can be used to back out the effective diffusivity across surfaces of constant ρ (aka isopycnals), since it controls the width (standard deviation) of these Gaussian curves. The article found that, unlike the constant diffusivity case presented above, κ varied by a factor of 20 as the water went through Drake Passage, which informed about the processes controlling said mixing. This type of work is critical, as diapycnal mixing is a key driver of the meridional overturning circulation. Despite the intricate physical processes controlling this mixing (often tied to internal waves breaking over rough bathymetry, such as that of Drake Passage), it is simply astounding that it can be described by the simplest of PDEs, the 1D diffusion equation, to such an accurate extent Fig. 3.3. This is a testimony to the power of math – even relatively accessible math!

[9] https://www.math.toronto.edu/courses/apm346h1/20129/L9.html

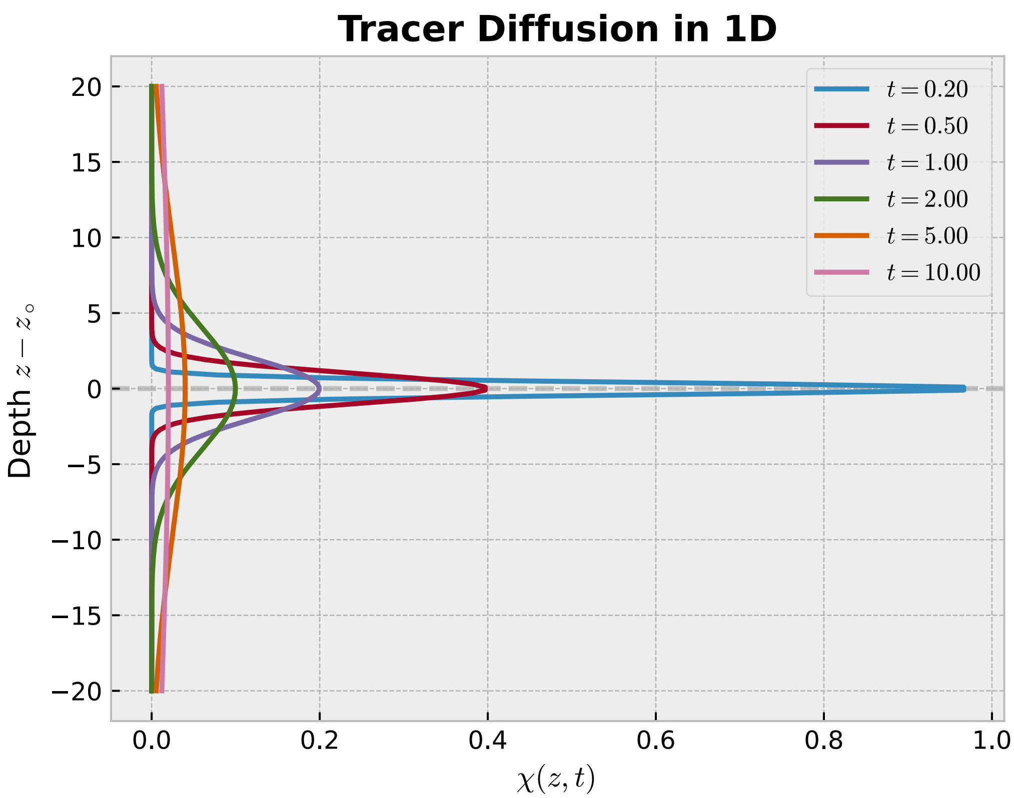

Figure 10: Evolution of a fictional tracer initially injected at depth $z_\circ$. Depth and time are non-dimensional here, with $D = 1$. Note that the area under each curve (the mass of the tracer) is the same in all cases, and equal to unity.

3.2 The Wave Equation

It may not surprise you to hear that the ocean is home to a lot of waves (Fig. 12). What may be surprising is that most of them are invisible to the naked eye, happening hundreds of meters below the surface, with only the faintest expression at the surface. There is a veritable zoo of waves: acoustic waves, capillary waves, external gravity waves, internal gravity waves, inertia-gravity waves (tides being the most famous example), Kelvin waves, Rossby waves, Yanai waves, and others.

A formal definition of a wave turns out to be surprisingly difficult ; all we'll say here is that they are periodic, traveling disturbances that can carry information across a medium. Wave-like behavior arises in a number of contexts: that is, there are many types of partial differential equations that admit wave-like solutions. However, the canonical wave equation is one that deserves special mention.

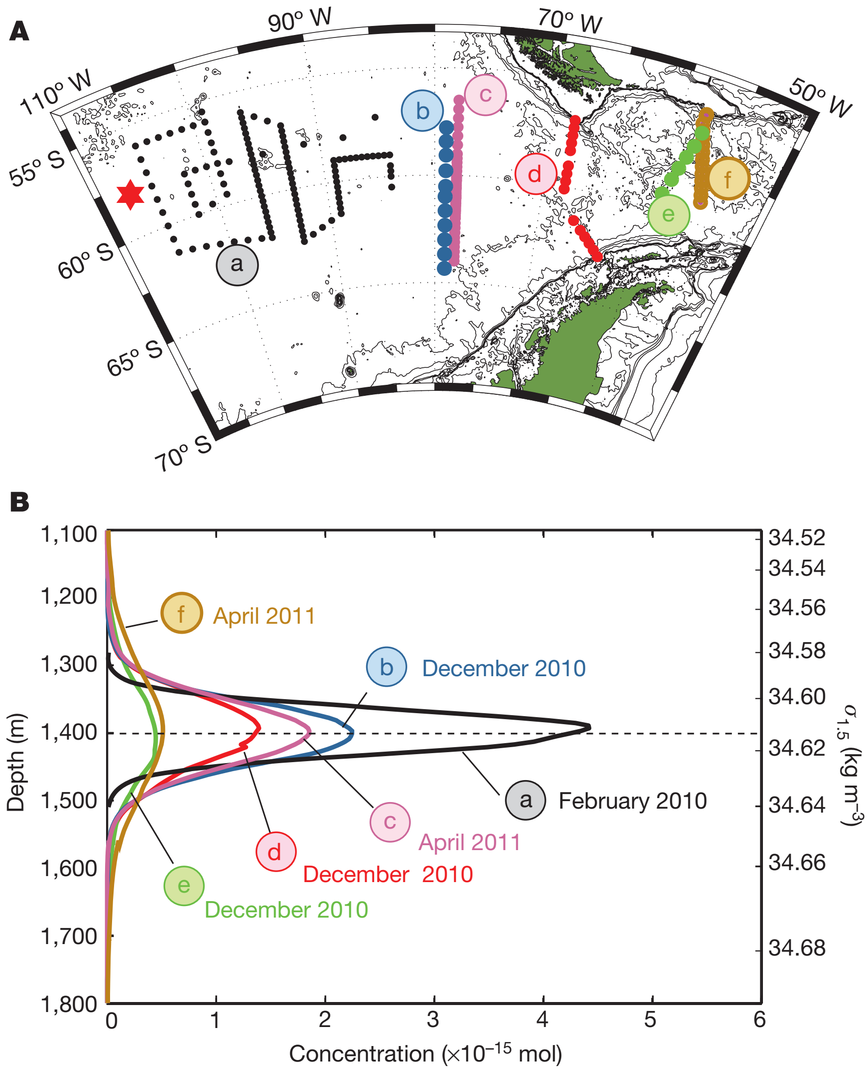

Figure 11: **Evolution of a real tracer injected at near 1500 m depth in the Southern Ocean**. Location of the tracer experiment and the vertical spread of the tracer during the first two years after release. A, Release in February 2009 (red star) and subsequent measurements and surveys: (a), East Pacific survey, one year after release; (b) and (c), sections near 78W, at 1.9 years and 2.2 years after release; (d), section at the western entrance to the Drake Passage, 1.9 years after release; (e) and (f), sections at the eastern exit of the Drake Passage, 1.9 and 2.2 years after release. B, Mean profiles obtained from each of these locations. These are plotted in potential density space (right-hand axis, $\sigma_{1.5}$, referenced to 1,500 dbar). Potential density is also translated into a depth scale (left-hand axis) using a mean densitydepth profile appropriate to the Drake Passage (the mean of sections (c) and (f)). Reproduced without any permission whatsoever from *Watson et al.* (2013).

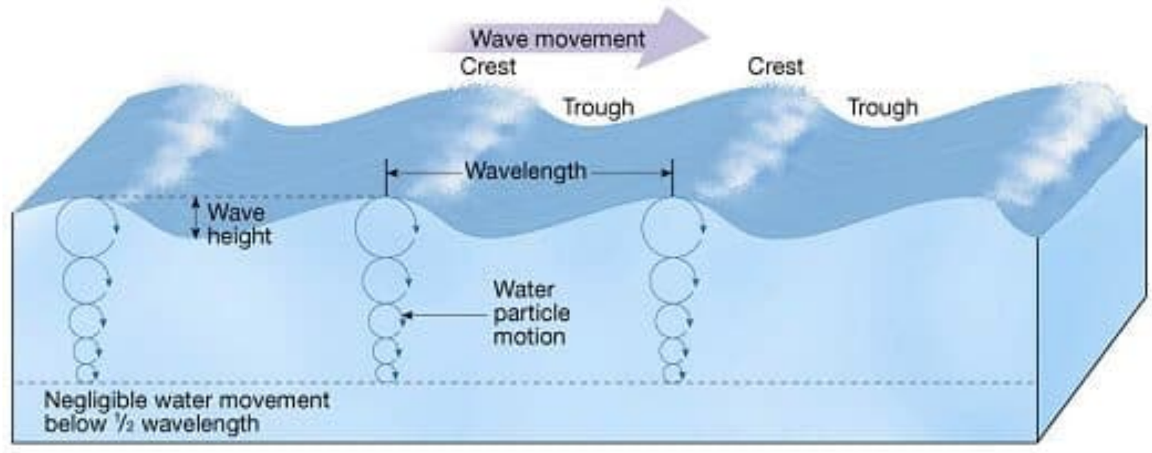

Figure 12: Depiction of surface waves, a form of **gravity waves**. While the waves can travel hundreds of kilometers, the motion of actual water particles is elliptical, with an amplitude that decays exponentially with depth.

3.2.1 General Form

The wave equation describes how perturbations in a given quantity $u(x, y, z, t)$ travel in space and time:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{3.5}$$

This is another linear, second-order PDE.

3.2.2 Oceanographic Application: the dispersion relation

It is customary to look for "plane wave" solutions to Eq. (3.5), of the form[10]:

$$u = \hat{u} \sin(\omega t - kx - ly)$$

where $\omega = \frac{2\pi}{T}$ is the angular frequency, $k = \frac{2\pi}{\lambda_x}$ is the zonal wavenumber, and $l = \frac{2\pi}{\lambda_y}$ is the meridional wavenumber ($\lambda_{x,y}$ are the zonal and meridional wavelengths of the waves, respectively; see Fig. 12). The great thing about sines and cosines is that they transform a PDE in terms of time and space into an algebraic equation involving frequency and wavenumbers. To see that, consider:

$$\frac{\partial u}{\partial t} = +\omega \hat{u} \cos(\omega t - kx - ly) \Rightarrow \frac{\partial^2 u}{\partial t^2} = -\omega^2 u$$

Similarly:

$$\frac{\partial u}{\partial x} = -k \hat{u} \cos(\omega t - kx - ly) \Rightarrow \frac{\partial^2 u}{\partial x^2} = -k^2 u$$

[10] A cosine would do just as well here

and

$$\frac{\partial u}{\partial y} = -l\hat{u}\cos(\omega t - kx - ly) \Rightarrow \frac{\partial^2 u}{\partial l^2} = -l^2 u$$

So the wave equation Eq. (3.5) becomes:

$$\omega^2 = c^2(k^2 + l^2) \tag{3.6}$$

Poof! No more derivatives! Just multiplications. Now that's something a computer can understand. It turns out that c is the *phase speed* of the waves, that it, the speed at which troughs and crests move, and may be expressed:

$$c = \frac{\omega}{\sqrt{k^2 + l^2}}$$

The more complicated notion of *group speed* $\boldsymbol{c_g}$ (the speed at which wave energy moves) is related to the gradient of ω in wavenumber space:

$$c_{gx} = \frac{\partial \omega}{\partial k} \tag{3.7}$$

$$c_{gy} = \frac{\partial \omega}{\partial l} \tag{3.8}$$

There is little point in dwelling on that math now without getting into the physics of waves. Just know that the wave equation describes certain kinds of wave in the ocean (e.g. acoustic waves), as well as an astonishing number of others, including seismic waves and electromagnetic waves. However, the ocean is also full of waves that do not obey Eq. (3.5), most notably Rossby waves and inertio-gravity waves.

3.3 Navier-Stokes equation

Finally, we have the fundamental equation of fluid dynamics, the Navier-Stokes equation. As we'll see in class, it fundamentally derives from Newton's second law – an expression of the conservation of linear momentum – and pertains to the velocity field:

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} \tag{3.9}$$

The equation writes:

$$\frac{d\boldsymbol{u}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \boldsymbol{g} + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} \tag{3.10}$$

where P is the pressure, ρ is the density, $\boldsymbol{g}$ is gravitational acceleration, and ν the kinematic viscosity in $\mathrm{m^2 s^{-1}}$. The term $\nu \boldsymbol{\nabla}^2 \boldsymbol{u}$ is an expression of how viscous forces dampen the flow (yes, even in water!), and its Laplacian form should remind you very strongly of the diffusion equation (Eq. (3.2)). Indeed, its role is to diffuse momentum, so left to its own devices, it acts to uniformize the velocity field, smoothing out any features. Eq. (3.10) originates in a simpler equation, due to Euler, which neglected viscous effects:

$$\frac{d\boldsymbol{u}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \boldsymbol{g} \tag{3.11}$$

Do not be deceived by its simplicity! In 1755 Euler wrote that while the physics were simple (Newton's second law), the mathematics had yet to be worked out. Nearly 270 years later, we still don't have a closed-form solution for that equation! The reason is that behind the innocuous acceleration term $\frac{d\boldsymbol{u}}{dt}$ (the rate of change of velocity) hides a term of baffling complexity. Indeed, recalling Eq. (2.10):

$$d\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x}dx + \frac{\partial \boldsymbol{u}}{\partial y}dy + \frac{\partial \boldsymbol{u}}{\partial z}dz + \frac{\partial \boldsymbol{u}}{\partial t}dt$$

What happens next may shock you: we divide both sides by dt:

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \boldsymbol{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \boldsymbol{u}}{\partial z}\frac{dy}{dt} + \frac{\partial \boldsymbol{u}}{\partial t}$$

However, by definition (Eq. (3.9)), $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$, so we can rewrite this, with a little re-arranging:

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y} + w\frac{\partial \boldsymbol{u}}{\partial z}$$

Here we further recognize that the last 3 terms are simply the dot product $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, so we arrive at:

$$\frac{d\boldsymbol{u}}{dt} = \underbrace{\partial_t \boldsymbol{u}}_{\text{Eulerian term}} + \underbrace{(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_{\text{Advective term}} \tag{3.12}$$

Eq. (3.12) is called the *material derivative*, because it follows the flow around. It is also called *Lagrangian derivative*[11], and this is what all the "Lagrangian floats" business is all about: following water parcels around. The first term is what a fixed (Eulerian) observer would see; a good example of that is a camera in a fixed frame watching the flow go by. The second term tracks variations along the flow, and is the reason for all our troubles: it is the essence of *nonlinearity*, meaning that slight perturbations in the velocity field can grow and interact in ways that are very hard to predict far into the future (chaos). To see this nonlinearity more clearly, recall Eq. (2.30):

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{u}|^2 + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u}$$

The first term is the gradient of the kinetic energy per unit mass, and is as nonlinear as it gets. The second also features a product of terms involving $\boldsymbol{u}$ and its spatial derivatives. Such nonlinear terms give rise to much of what is interesting about fluid motion – but is also a major pain to solve. It is the main reason why the Clay Institute's Millennium Prize about this equation exists – and its still unclaimed. And it is the main reason weather and climate prediction are so uncertain.

[11] After Joseph-Louis Lagrange

We may thus re-write Eq. (3.10) to show its bare teeth and make its dangers plain:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \boldsymbol{g} + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} \tag{3.13}$$

Together with other conservation principles for mass, enthalpy, and salt, as well as the all-important equation of state (Eq. (2.2.2)), the Navier-Stokes equation forms the cornerstone of our current understanding of oceanic motion. Because no-one alive can solve this equation, be reassured that you never will be asked to!

In the rest of the class, we will first add a major twist to this equation (Earth's rotation), then deploy treasures of ingenuity and laziness to avoid solving it explicitly. Instead, we will ruthlessly neglect terms until we get obtain tractable solutions. To do this, we'll have to apply physical intuition, so this falls outside the purview of this little mathematical recreation. Come to class for some physics!

Conclusion

We've seen how the basic notion of differentiation (extended to multiple variables) can be used to formulate equations that govern many aspects of ocean physics. Because the ocean is a 3D medium, these equations also involve a fair amount of geometry, specifically the gradient and its dot and cross products with other vectors. This forms the mathematical ABC necessary to parse a PO textbook, but we would be remiss in not mentioning that oceanography can and does involve deeper mathematics. For instance, at a higher level, one needs to master singular perturbation theory (*Pedlosky*, 2013), linear algebra (*Wunsch*, 1996) and optimization (*Wunsch*, 2006). Perhaps more philosophically, the fact that the basic equations of fluid dynamics are so difficult to solve suggests the possibility that they are not quite the right framework ; it is possible than in the future some new mathematical framework will come along that will make us laugh very hard at the unsolvable Navier-Stokes equations, and provide more natural methods to solve for the flow. In the meantime, you'll have to contend with a bit of vector calculus, so better get used to it!

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